Logic for Linguists:

Lecture 8

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Possible Worlds in Linguistics

Let's consider the following two sentences:

1. It's certain that you will find a job, and it's conceivable that it will be a good-paying one.
2. It's conceivable that you'll find a job, and it's certain that it will be a good-paying one.

A natural semantics seems to require what McCawley calls "World Creating Predicates".

We might model (1) by having one world w, the present world, and many possible worlds in the near future, some of which you have a good paying job, and some in which you don't.

We could model (2) similarly, but in this case it's the question of whether you will have a job that is up in the air.

Applications in Linguistics

Partee describes six broad areas where “possible worlds” arise in linguistics:

- The identification of propositions with sets of possible worlds.
- The analysis of intensional phenomena with functions from possible worlds to their extensions.
- The semantics of propositional attitudes.
- The semantics of conditionals.
- The semantics of questions and the pragmatics of the question-answer relations.
- Pragmatics in general, and presuppositions in particular.

For more see *Handbook for Modal Logic*, Chapter 19: Applications of Modal Logic in Linguistics.

There is a free version available here:
`https://iulg.sitehost.iu.edu/moss/linguistics.pdf`

Reasoning about Possible Worlds is Hard

A sentence in natural language may describe complex relationships between possible worlds. For example, instead of saying "certain" we might have said "more likely than not".

A simple sentence or argument involving just necessity and possibility might be hard to follow and quickly become complex.

Let's consider the following argument for theological fatalism:

- Necessarily if God foreknows X, then X will happen.
- God foreknows X.
- Therefore necessarily X will happen.

Is this argument valid?

Modal Logic

Modal Logic to the rescue! This logic allows us to talk about necessary and possible truths.

There are lots of modal logics floating around. We will discuss some simple modal logics.

The important point is that modal logic introduces two operators:

- Necessity: $\Box$ – $\Box A$ denotes that A is neccesary true or, put another way, true in every possible world.
- Possibility: $\Diamond$ – $\Diamond A$ denotes that A is possibly true or that A is true in some world reachable from the current world.

System $\mathbb{K}$

This is perhaps the simplest modal logic discussed, and it's from this logic that we build up the more complex ones.

We recall that propositional formulas are those formed from variables closed under taking Boolean combinations. For example:

$$A,\ A \vee B,\ A \rightarrow B,\ (A \leftrightarrow B) \wedge C$$

The formulas of $\mathbb{K}$ are defined similarly, except now we allow for two additional operators: $\Box$ and $\Diamond$. More formally:

- all propositional variables are formulas of system $\mathbb{K}$ and
- if ϕ and ψ are formulas of system $\mathbb{K}$ then $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$, $\psi \rightarrow \phi$, $\phi \leftrightarrow \psi$, $\Box\phi$, and $\Diamond\phi$, are all formulas of system $\mathbb{K}$.

Note: I have introduced $\Box$ and $\Diamond$ as two separate operators for ease of understanding. From now on we consider $\Diamond\phi$ to be shorthand for $\neg\Box\neg\phi$.

Examples

We have not yet formally defined the semantics, but this is all getting a bit “symbol heavy‘”’ so let’s consider some examples.

Example 1:

- “It’s certain that you will find a job, and it’s conceivable that it will be a good-paying one.”
- $\Box A \wedge \Box(A \rightarrow B)$

Example 2:

- “It’s conceivable that you’ll find a job, and it’s certain that it will be a good-paying one.”
- $\Diamond A \wedge \Box(A \rightarrow B)$

Theorems in $\mathbb{K}$

We use the word theorem in logic to mean those statements that are always true in a given system.

Recall that the theorems of propositional logic are the tautologies. We define the theorems of $\mathbb{K}$ by adding the following rules to the logical rules for propositional logic:

- K1: (Necessitation Rule) If ϕ is a theorem of $\mathbb{K}$ then $\Box\phi$ is a theorem of $\mathbb{K}$.
- K2: (Distribution Axiom) for all $\mathbb{K}$ formulas ϕ and ψ the following is a theorem $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$.

Examples

Let's consider an example we often return to:

"If bob is from California, and if bob is from California then he is from America, then bob is from America."

This is an instance of the propositional tautology:

$$\phi := (A \land (A \to B)) \to B$$

and so is a theorem of System $\mathbb{K}\dot{T}$hen

$$\Box((\Box A \land (\Box A \to \Diamond B)) \to \Diamond B)$$

is a theorem of System $\mathbb{K}$.

A Modal Family

The trouble is that modal logic is not really one thing. Which axioms we want to accept really depends on what sort of reasoning we are trying to model.

We might want to reason about...

- ethics, and define operators that denote obligation, permission, and prohibition (deontic logic);
- time, and define operators that denote that something will always be the case, may be the case in the future, has always been the case, etc. (temporal logic); or
- knowledge, and define operators that denote that an agent always believes, may believe, etc. (epistemic logic).

We can formalise each of these logics as modal logics by picking a suitable set of axioms. For this reason there are lots of modal logics developed each designed with different applications (or families of applications) in mind.

An Example

Let's consider the following:

$$\Box A \rightarrow A$$

It turns out you cannot establish that this is a tautology within System $\mathbb{K}$, so if we want it we would need to include it as an axiom. So should we?

- Well, on the one hand it seems very reasonable. If it is Necessarily true that A is true then surely A is true?
- On the other hand, if we are trying to model ethics and $\Box$ is supposed to denote "it ought to be the case", then this axiom would be ridiculous.

Semantics: Kripke Frames

We have been reasoning very informally. It's time to give a formal semantics to modal formulas.

This is done using Kripke frames. Intuitively, a (Kripke) frame is a set of possible worlds and a set of arrows such that there is an arrow from world w_1 to world w_2 if w_2 can be reached immediately from w_1.

More formally, a frame F is a pair (W, R), where

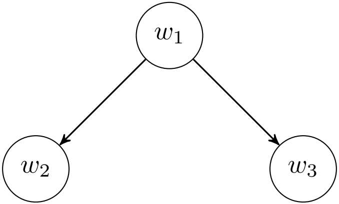
- W is a set, we call the set of worlds and
- $R \subseteq W \times W$.

Frames as Graphs

Let's consider the frame $F = (W, R)$, where

- $W = \{w_1, w_2, w_3\}$ and
- $R = \{(w_1, w_2), (w_1, w_3)\}$.

This can be summarised in the following diagram:



So in this frame from world w_1 we can arrive at either world w_2 or world w_3.

What is in a World?

But we want much more! We also want to know things about each of these worlds. In which worlds do we have a job? In which worlds are those high paying?

We now introduce a function that associates with each proposition the set of worlds for which that proposition is considered true.

Let Φ_0 be the set of all atomic propositions (e.g. $A, B, \ldots$). Let $F = (W, R)$ be a frame. Let π be a function that takes in an atomic proposition in Φ_0 and outputs a subset of W.

The intuition is that for any $p \in \Phi_0$ and $w \in W$ we have that $w \in \pi(p)$ if, and only if, p is considered to be true in w.

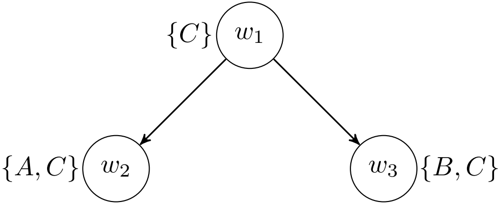
Updating our Example...

Let's consider the frame $F = (W, R)$, where

- $W = \{w_1, w_2, w_3\}$ and
- $R = \{(w_1, w_2), (w_1, w_3)\}$.

Let's define π such that $\pi(A) = \{w_2\}$, $\pi(B) = \{w_3\}$, and $\pi(C) = \{w_1, w_2, w_3\}$.

We now have the following diagram for (F, π):



Let's get Formal

A model $\mathcal{M}$ is a frame $F = (W, R)$ and a truth function π. We often write $\mathcal{M} = (F, \pi)$ or $\mathcal{M} = (W, R, \pi)$.

We often draw models just was we did on the previous slide as graphs (nodes with arrows between them) where each possible world is labelled by the set of atomic propositions true in that world.

We now define what it means for a modal formula to be true in some world w within a model.

This should just be formalising the intuition we have been using all talk...

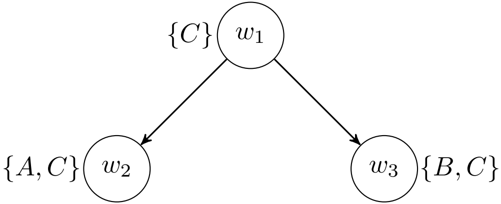
Semantics

Let $\mathcal{M} = (W, R, \pi)$ be a model and let $w \in W$. We define the truth predicate $\Vdash$ for modal logic. We say that:

- $\mathcal{M}, w \Vdash p$ iff $w \in \pi(p)$ for any $p \in \Phi_0$
- $\mathcal{M}, w \Vdash \neg\phi$ iff $\mathcal{M}, w \nVdash \neg\phi$
- $\mathcal{M}, w \Vdash \phi \wedge \psi$ iff $\mathcal{M}, w \Vdash \phi$ and $\mathcal{M}, w \Vdash \psi$
- $\mathcal{M}, w \Vdash \phi \vee \psi$ iff $\mathcal{M}, w \Vdash \phi$ or $\mathcal{M}, w \Vdash \psi$
- $\mathcal{M}, w \Vdash \Box\phi$ iff for every $u \in W$ such that $R(w, u)$ we have $\mathcal{M}, u \Vdash \phi$
- $\mathcal{M}, w \Vdash \Diamond\phi$ iff there exists $u \in W$ such that $R(w, u)$ and $\mathcal{M}, u \Vdash \phi$

Whew...now onto Examples!

Let's return to our favourite example:



Which of the following are true:

- $\mathcal{M}, w_1 \Vdash \Diamond A$,
- $\mathcal{M}, w_2 \Vdash \Diamond C$,
- $\mathcal{M}, w_1 \Vdash (\Diamond A) \wedge (\Diamond B)$,
- $\mathcal{M}, w_1 \Vdash \Diamond (A \wedge B)$, and
- $\mathcal{M}, w_1 \Vdash \Diamond \Box \Diamond (A \vee \neg A)$.

Some More Terminology

Let's introduce some standard terminology:

- If $\mathcal{M}, w \Vdash \phi$ we say that ϕ holds in $\mathcal{M}$ in world w.
- We write $\mathcal{M} \Vdash \phi$, and say that ϕ is globally true in $\mathcal{M}$, if for all $w \in W$, $\mathcal{M}, w \Vdash \phi$.
- We say that ϕ is satisfiable in $\mathcal{M}$ if there exists $w \in W$ such that $\mathcal{M}, w \Vdash \phi$.

Does this Semantics Make Sense?

Are the theorems of system $\mathbb{K}$ tautologies? That is to say, are they globally true in every model? We recall that all propositional tautologies are theorems of system $\mathbb{K}$, and there are two other rules:

- K1: (Necessitation Rule) If ϕ is a theorem of $\mathbb{K}$ then $\Box\phi$ is a theorem of $\mathbb{K}$.
- K2: (Distribution Axiom) for all $\mathbb{K}$ formulas ϕ and ψ the following is a theorem $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$.

It is easy to see that all propositional tautologies are globally true in every model. Moreover, if ϕ is globally true in every model then clearly $\Box\phi$ is as well. It is similarly easy to show that $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ is always a theorem.

Let's Return to the Argument

The argument:

- Necessarily if God foreknows X, then X will happen.
- God foreknows X.
- Therefore necessarily X will happen.

So that would be formalised as:

$$(\Box(P \to Q) \land P) \to \Box Q$$

Exercise 1: Construct a model in which this fails, and hence show that this is not a theorem of system $\mathbb{K}$. It follows that the argument is not valid.

Exercise 2: Replace the conclusion with "X will happen" and formalise the argument in symbols again. Is this now a theorem of system $\mathbb{K}$?

The Ontological Argument for God

Ax. 1.	$(P(\varphi) \wedge \Box\, \forall x(\varphi(x) \Rightarrow \psi(x))) \Rightarrow P(\psi)$
Ax. 2.	$P(\neg\varphi) \Leftrightarrow \neg P(\varphi)$
Th. 1.	$P(\varphi) \Rightarrow \Diamond\, \exists x\, \varphi(x)$
Df. 1.	$G(x) \Leftrightarrow \forall\varphi(P(\varphi) \Rightarrow \varphi(x))$
Ax. 3.	$P(G)$
Th. 2.	$\Diamond\, \exists x\, G(x)$
Df. 2.	$\varphi \text{ ess } x \Leftrightarrow \varphi(x) \wedge \forall\psi\, (\psi(x) \Rightarrow \Box\, \forall y(\varphi(y) \Rightarrow \psi(y)))$
Ax. 4.	$P(\varphi) \Rightarrow \Box\, P(\varphi)$
Th. 3.	$G(x) \Rightarrow G \text{ ess } x$
Df. 3.	$E(x) \Leftrightarrow \forall\varphi(\varphi \text{ ess } x \Rightarrow \Box\, \exists y\, \varphi(y))$
Ax. 5.	$P(E)$
Th. 4.	$\Box\, \exists x\, G(x)$

What's Next?

There are so many other modal logics to which we have already alluded. Let me mention just two that comes up very often in application:

We define S4 by adding the following axioms to $\mathbb{K}$:

- M: $\Box\phi \rightarrow \phi$
- 4: $\Box\phi \rightarrow \Box\Box\phi$

We define $S5$ by adding the following axiom to S4:

- 5: $\Diamond\phi \rightarrow \Box\Diamond\phi$.

This logic is often used to reason about knowledge! The above axioms in that context can be understood as stating:

- M: if I know something then it's true
- 4: if I know something then I know that I know it.
- 5: if I don't know something then I know that I don't know it.

Temporal Logic

We can define a modal logic for reasoning about time by putting some restrictions on the definition of a frame and adding corresponding axioms.

Exactly what restrictions you put on the frame depend on the application you have in mind.

It is common to restrict (W, R) such that R is a total order on W (this means that $(a, b) \in R$ can be coherently interpreted as a coming before b).

We might also require that R is dense (i.e. If $a < c$ then there exists b such that $a < b < c$).

Summary

This lecture included a lot! Let's remind ourselves what we discussed.

- We started by discussing how "world creating predicates" arise often in natural language and philosophical argumentation.
- We noted that these predicates can be hard to understand.
- We introduced modal logic and system $\mathbb{K}$.
- We talked about Kripke frames.
- We defined the notion of a model.
- We defined the semantics of modal logic.
- We briefly discussed extensions of $\mathbb{K}$.

Reading Material

This is a wonderful discussion of modal logic in linguistics (this is an online version of a chapter from the *Handbook of Modal Logic*):

- https://iulg.sitehost.iu.edu/moss/linguistics.pdf

Here is a great resource introducing modal logic and some very fun puzzles that we can use modal logic to resolve:

- https://www.logic.at/lvas/185249/MuddyChildren-WiseMen_4in1.pdf

Here is a broad introduction of modal logic:

- https://plato.stanford.edu/entries/logic-modal/

Some standard textbooks:

- B.F. Chellas. *Modal Logic: An Introduction.*
- G. Hughes and M.J. Cresswell. *A New Introduction to Modal Logic.*
- P. Blackburn, M. De Rijke, and Y. Venema. *Modal Logic.*
- P. Blackburn, J. van Benthem, and F. Wolter. *Handbook of Modal Logic.*