Logic for Linguists:

Lecture 4

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The Talk Today

In this lecture we will discuss generalised quantifiers. In particular, we will:

- discuss some of the early history of logic and quantification,
- talk about what it means for two models to be equivalent,
- introduce generalised quantifiers,
- discuss some natural applications in linguistics,
- use these applications to motivate further developing our framework for generalised quantifiers, and finally
- we will discuss some general theory for extensions of first-order logic.

Syllogisms

The most famous achievement of Aristotle in logic is his development of syllogistic inference. A syllogism is an argument of a very specific sort, consisting of two premises (major and minor) and one conclusion. The standard example:

> "All men are mortal. Socrates is a man. Therefore, Socrates is mortal."

We have

- Major Premise: "All men are mortal."
- Minor Premise: "Socrates is a man."
- Conclusion: "Socrates is mortal."

Another example:

"Everything white is sweet. Salt is white. Therefore, salt is sweet."

Importantly, the premises and conclusion have structure. Each is formed from two terms, where a term is a subjects or predicate (e.g. "salt", "white", "sweet").

The premises are required to have one term in common (called the middle term) and the conclusion is required to only use the remaining terms.

Generalised Quantifiers

Aristotle considered four different quantifiers: “all”, “no”, “some”, and “not all”.

Let’s pause for some history. In the 1870s Gottlob Frege introduced the first predicate calculus. One of his most important results established that each of these quantifiers can be expressed in predicate logic. In particular:

- “All A are B” can be written as $\forall x(A(x) \rightarrow B(x))$,
- “No A is a B” can be written as $\neg\exists x(A(x) \wedge B(x))$,
- “Some A are B” can be written as $\exists x(A(x) \wedge B(x))$, and
- “Some A is not B” can be written as $\exists x(A(x) \wedge \neg B(x))$.

In this sense predicate logic superseded syllogistic reasoning. Can we introduce generalised quantifiers for predicate logic? Can we generalise the ordinary universal and existential quantifiers?

When Are Two Models the Same?

In order to talk formally about generalised quantifiers we first need to introduce another key idea, the notion of an isomorphism.

An isomorphism is a function between two models that witnesses the fact that these two models are essentially the same up to presentation. Before we define this concept rigorously let's consider a few concrete examples. Let τ be the vocabulary consisting of

- Constant Symbols: a ("Alice"), b ("Bob"), and c ("Charlie")
- Unary Relation Symbol: L ("is a Linguist")
- Binary Relation Symbol: F ("is friends with")

Three Examples

Let $\mathcal{M}$ be the τ-model with universe $\{1, 2, 3\}$ and

- $a^{\mathcal{M}} = 1$, $b^{\mathcal{M}} = 2$, and $c^{\mathcal{M}} = 3$;
- $L^{\mathcal{M}} = \{1, 2\}$; and
- $F^{\mathcal{M}} = \{(1, 2)(2, 1)\}$.

Let $\mathcal{N}$ be the τ-model with universe $\{2, 3, 4\}$ and

- $a^{\mathcal{N}} = 2$, $b^{\mathcal{N}} = 3$, and $c^{\mathcal{N}} = 4$;
- $L^{\mathcal{N}} = \{2, 3\}$; and
- $F^{\mathcal{N}} = \{(2, 3)(3, 2)\}$.

Let $\mathcal{P}$ be the τ-model with universe $\{1, 2, 3, 4\}$ and

- $a^{\mathcal{P}} = 2$, $b^{\mathcal{P}} = 3$, and $c^{\mathcal{P}} = 4$;
- $L^{\mathcal{P}} = \{2, 3\}$; and
- $F^{\mathcal{P}} = \{(2, 3)(3, 2)\}$.

Which of these models are the same?

Bijection

Let A and B be two sets. We say that $f : A \to B$ is a bijection if

- for every $b \in B$ there is some $a \in A$ such that $f(a) = b$ and
- for every $a_1, a_2 \in A$ if $f(a_1) = f(a_2)$ then $a_1 = a_2$.

Isomorphism

Let τ be a vocabulary. Let $\mathcal{M}$ and $\mathcal{N}$ be τ-models with universes M and N, respectively. We say that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if there exists a bijection $f : M \to N$ such that

- for every constant symbol $c \in \tau$ we have $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$,
- for every relation symbol $R \in \tau$ with arity r and every $a_1, \ldots, a_r \in M$ we have

$$R^{\mathcal{M}}(a_1, \ldots, a_r) \text{ if, and only if, } R^{\mathcal{N}}(f(a_1), \ldots, f(a_r)), \text{ and}$$

- for every function symbol $g \in \tau$ with arity r every $a_1, \ldots, a_r \in M$ we have

$$f(g^{\mathcal{M}}(a_1, \ldots, a_r)) = g^{\mathcal{N}}(f(a_1), \ldots, f(a_r)).$$

So, returning to the previous examples, which of them are the same?

Example

Let $\mathcal{M}$ be the τ-model with universe $\{1, 2, 3\}$ and

- $a^{\mathcal{M}} = 1$, $b^{\mathcal{M}} = 2$, and $c^{\mathcal{M}} = 3$;
- $L^{\mathcal{M}} = \{1, 2\}$; and
- $F^{\mathcal{M}} = \{(1, 2)(2, 1)\}$.

Let $\mathcal{N}$ be the τ-model with universe $\{2, 3, 4\}$ and

- $a^{\mathcal{N}} = 2$, $b^{\mathcal{N}} = 3$, and $c^{\mathcal{N}} = 4$;
- $L^{\mathcal{N}} = \{2, 3\}$; and
- $F^{\mathcal{N}} = \{(2, 3)(2, 3)\}$.

Let $\mathcal{P}$ be the τ-model with universe $\{1, 2, 3, 4\}$ and

- $a^{\mathcal{P}} = 2$, $b^{\mathcal{P}} = 3$, and $c^{\mathcal{P}} = 4$;
- $L^{\mathcal{P}} = \{2, 3\}$; and
- $F^{\mathcal{P}} = \{(2, 3)(2, 3)\}$.

Which of these models are the isomorphic?

Isomorphism $\rightarrow$ Logic Equivalence

The crucial point for us as logicians is that if two structures are isomorphic they are logically indistinguishable. In other words, if $\mathcal{M}$ and $\mathcal{N}$ are structures over some vocabulary τ and are isomorphic then for every first-order sentence ϕ over τ we have that

$$\mathcal{M} \models \phi \text{ if, and only if, } \mathcal{N} \models \phi.$$

We will want to define generalise quantifiers in such a way that a quantifier cannot distinguish between isomorphic objects either.

We say that a class of τ-structures $\mathcal{C}$ is closed under isomorphism if for every structure $\mathcal{M} \in \mathcal{C}$ and every τ-structure $\mathcal{N}$ if $\mathcal{M}$ is isomorphic to $\mathcal{N}$ then $\mathcal{N} \in \mathcal{C}$.

A Review of Quantifiers

Before we introduce generalised quantifiers let's first revisit out usual neighbourhood quantifiers. Consider the formulas

$$\forall x \psi(x) \text{ and } \exists x \psi(x)$$

Syntactically these formulas are just applications of quantifiers that bind a single variable x in a single formula ψ.

Let $\mathcal{M}$ be a model. We have that

- $\mathcal{M} \models \forall x \psi(x)$ if, and only if, $\{a \in M : \mathcal{M} \models \psi[a]\} = M$, and
- $\mathcal{M} \models \exists x \psi(x)$ if, and only if, $\{a \in M : \mathcal{M} \models \psi[a]\} \neq \emptyset$.

Type (1) Generalised Quantifiers

We say that Q is a generalised quantifier of type (1) if

- The Syntax: Q is an operator that binds a single variable in a single formula. In other words if ψ is a formula then $Qx\psi$ is a formula in which all free occurrences of x in ψ have been bound; and
- The Semantics: there is some isomorphism-closed set of structures $\mathcal{C}_Q$ over the vocabulary with a single unary relation symbol and for every formula of the form $Qx\psi$ and every structure $\mathcal{M}$ in the vocabulary of ψ we have that

$$\mathcal{M} \models Qx\psi \text{ if, and only if, } (M, \{a \in M : \mathcal{M} \models \psi[a]\}) \in \mathcal{C}_Q.$$

We define the extension of first-order logic by the quantifier Q, which we denote by FO(Q), as we defined FO, but this time we allow $\forall$, $\exists$, and Q as quantifiers.

Examples

Let's look at a few examples. In order to define a generalised quantifier it suffices to define the corresponding class of structures.

We define the quantifier $\exists^{>2}$ by letting

$$\mathcal{C}_{\exists^{>2}} := \{(M, A) : M \text{ a set}, A \subseteq M, |A| > 2\}$$

Let $\mathcal{N}$ be a model. Let ψ be a first-order formula with a single free variable x. We have:

$$\begin{aligned} \mathcal{N} \models \exists^{>2} x \psi \text{ iff } & (N, \{a \in N : \mathcal{N} \models \psi[a]\}) \in \mathcal{C}_{\exists^{>2}} \\ \text{iff } & |\{a \in N : \mathcal{N} \models \psi[a]\}| > 2 \\ \text{iff } & \text{ there are more than two different elements in } \mathcal{N} \text{ that satisfy } \psi. \end{aligned}$$

More Examples

We define the quantifier Q_{inf} by letting

$$\mathcal{C}_{Q_{\text{inf}}} := \{(M, A) : M \text{ a set, } A \subseteq M, \ A \text{ is infinite}\}$$

We define the quantifier Q_{odd} by letting

$$\mathcal{C}_{Q_{\text{odd}}} := \{(M, A) : M \text{ a set, } A \subseteq M, \ A \text{ is finite and } |A| \text{ is odd}\}$$

We define the quantifier Q_{maj} by letting

$$\mathcal{C}_{Q_{\text{maj}}} := \{(M, A) : M \text{ a set, } A \subseteq M, \ A \text{ is finite and } |A| > \frac{|M|}{2}\}$$

Application to Natural Language

In the 1960s Richard Montague noted that noun phrases can be identified with subsets of some domain. In the 1980s linguists and logicians working on model-theoretic approaches with generalised quantifiers to natural language semantics greatly expanded on this observation.

Example: "Some linguists study."

The noun phrase "some linguists", which consists of a determiner ("some") and a noun ("linguists") can be identified with those subsets of our domain that contain "some linguists". In other words, we can identify this noun phrase with the generalised quantifier Q_{SL} defined by letting

$$\mathcal{C}_{Q_{\mathrm{SL}}} := \{(M, A) : M \text{ a set}, A \subseteq M, A \text{ contains at least one linguist}\}$$

A Natural Criticism

There are somethings very unsatisfying about this approach. We have now a generalised quantifier that corresponds to "some linguists", but this bundles together the determiner in such a way that we cannot make sense of it without referring directly to the universe in question.

This seams wrong. Surely the meaning of expressions such as "most", "always", "every", etc. do not depend on the particular objects we are considering?

We need a more general generalised quantifier.

Generalised Quantifiers

Let $\tau := \{R_1, \ldots, R_t\}$ be a vocabulary consisting of only relations and let r_i be the arity of R_i. A type $(r_1, \ldots, r_t)$ generalised quantifier Q is defined as follows.

- Syntactically: Q is an operator that binds the variables $\vec{x}_1, \ldots, \vec{x}_t$, where each $\vec{x}_i$ is a tuple of r_i variables, and such that if $\psi_1, \ldots, \psi_t$ are formulas then

$$Q\vec{x}_1, \ldots, \vec{x}_t(\psi_1, \ldots, \psi_t)$$

  is a formula.

- Semantically: there is some isomorphism-closed set $\mathcal{C}_Q$ of τ-models such that for every model $\mathcal{M}$ and formulas $\psi_1, \ldots, \psi_t$

$$\mathcal{M} \models Q\vec{x}_1, \ldots, \vec{x}_t(\psi_1, \ldots, \psi_t) \text{ iff } (M, \psi_1^{\mathcal{M}}, \ldots, \psi_t^{\mathcal{M}}) \in \mathcal{C}_Q,$$

  where

$$\psi_i^{\mathcal{M}} := \{\vec{a} \in M^{r_i} : \mathcal{M} \models \psi_i[\vec{a}]\}.$$

Back to Natural Language

We can now model the application of a determiner using $(1,1)$ generalised quantifiers.

Earlier Example: "Some linguists study."

Recall, a $(1,1)$ generalised quantifier is defined over the vocabulary $\{R_1, R_2\}$, where both relations are unary relation symbols.

Let us define Q_{some} defined by letting

$$\mathcal{C}_{Q_{\text{some}}} := \{(M, A, B) : A, B \subseteq M, A \cap B \neq \emptyset\}.$$

Now let $\phi_L(x)$ and $\phi_S(x)$ be formulas such that for a model $\mathcal{M}$ with some appropriate universe (say the set of all people) we have

$$\text{for every } a \in M,\ a \text{ is a linguist iff } \mathcal{M} \models \psi_L[a] \text{ and}$$
$$\text{for every } a \in M,\ a \text{ studies iff } \mathcal{M} \models \psi_S[a].$$

The above example then corresponds to $Q_{\text{some}} x_1 x_2(\phi_L, \phi_S)$. We can handle many other determiners and similar expressions using generalised quantifiers.

Comparing Quantifiers

We have all of these quantifiers floating around. We can use them to define different extensions of first-order logic. We might be interested in comparing these extensions and understanding which are more expressive than others.

What do we mean by "more expressive"?

We write $\mathrm{FO}(\vec{Q}_1) \leq \mathrm{FO}(\vec{Q}_2)$ and say that $\mathrm{FO}(\vec{Q}_1)$ is at most as expressive as $\mathrm{FO}(\vec{Q}_2)$ if for every vocabulary τ and every formula ϕ_1 in $\mathrm{FO}(\vec{Q}_1)$ over τ there is some formula ϕ_2 in $\mathrm{FO}(\vec{Q}_2)$ over τ such that $\mathrm{mod}(\phi_1) = \mathrm{mod}(\phi_2)$.

If $\mathrm{FO}(\vec{Q}_1) \leq \mathrm{FO}(\vec{Q}_2)$ and $\mathrm{FO}(\vec{Q}_2) \leq \mathrm{FO}(\vec{Q}_1)$ we say that these two logics are equally expressive and write $\mathrm{FO}(\vec{Q}_2) \equiv \mathrm{FO}(\vec{Q}_1)$.

Example

Recall the quantifier $\exists^{>2}$ we defined earlier. We now show that $\mathrm{FO}(\exists^{>2}) \equiv \mathrm{FO}$. Let ϕ be a formula in $\mathrm{FO}(\exists^{>2})$ of the form

$$\phi := \exists^{>2} x \psi,$$

where ψ is a FO-formula. Let

$$\phi' := \exists y_1 \exists y_2 \exists y_3 [(y_1 \neq y_2 \wedge y_2 \neq y_3 \wedge y_1 \neq y_3) \wedge (\exists x(x = y_1 \wedge \psi)) \wedge (\exists x(x = y_2 \wedge \psi)) \wedge (\exists x(x = y_3 \wedge \psi))].$$

Note that ϕ' is a first-order formula over the same vocabulary as ϕ. It can be seen that $\mathrm{mod}(\phi) = \mathrm{mod}(\phi')$. From this we can show that $\mathrm{FO}(\exists^{>2}) \equiv \mathrm{FO}$.

Lindström and the Limits of First-Order Logic

We spoke last week about completeness and how completeness is in some sense the defining feature of first-order logic. I cannot state this formally, but I'll state a slightly incorrect version of this result.

Theorem (Not Quite Lindström's Theorem.)

If L is complete then $L \leq \mathrm{FO}$.

This is the formal version for interested listeners

Theorem (Lindström's Theorem.)

If L is compact, has the Löwenheim property, and $\mathrm{FO} \leq L$, then $\mathrm{FO} \equiv L$.

Summary

In this lecture we

- introduced syllogisms and discussed some history of quantification in logic,
- introduced the central notion of two structures being isomorphic,
- we formally defined the notion of a type (1) generalised quantifier,
- we discussed some applications to formal linguistics,
- motivated by these applications we introduced a more general framework for generalised quantifiers, and
- we discussed how to compare extensions of first-order logic and concluded with Lindström's theorem.

Some Reading (1)

This is a wonderful all-around introduction to generalised quantifiers in lingustics in five lectures:

- Lecture 1: http://www.math.helsinki.fi/logic/sellc-2010/course/GQ-Guangzhou1.pdf
- Lecture 2: http://www.math.helsinki.fi/logic/sellc-2010/course/GQ-Guangzhou2.pdf
- Lecture 3: http://www.math.helsinki.fi/logic/sellc-2010/course/GQ-Guangzhou3.pdf
- Lecture 4: http://www.math.helsinki.fi/logic/sellc-2010/course/GQ-Guangzhou4.pdf
- Lecture 5: http://www.math.helsinki.fi/logic/sellc-2010/course/GQ-Guangzhou5.pdf

Some Reading (2)

For those interested in syllogisms:

- `https://plato.stanford.edu/entries/aristotle-logic/`

For those interested in further reading on generalised quantifiers in linguistics:

- `https://www.jstor.org/stable/25001052?seq=1` (check this one out!)
- `https://www.cl.cam.ac.uk/~aac10/teaching-notes/gq.pdf` (these are lecture notes, very accessible)
- `https://www.springer.com/gp/book/9780792331292`