Logic for Linguists Example Sheet 1

Dr. Gregory Wilsenach

Propositional Logic

1. Write out the truth tables for $A \wedge B$, $A \vee B$, $A \rightarrow B$, and $A \leftrightarrow B$ (recall $A \leftrightarrow B$ is shorthand for $(A \rightarrow B) \wedge (B \rightarrow A)$). See Section 2 of `https://www.cl.cam.ac.uk/teaching/1112/LogicProof/logic-notes.pdf` in order to check yourself.

2. Define what it means for a propositional sentence to be consistent (i.e. satisfiable), inconsistent, or valid (i.e. a tautology).

3. For each of the following formulas write out the truth table and state whether the formula is valid or at least satisfiable:

   (a) $A \wedge B \rightarrow A$

   (b) $(\neg(\neg A)) \leftrightarrow A$

   (c) $(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$

   (d) $A \rightarrow \neg A$

   (e) $(\neg A \wedge \neg B) \leftrightarrow \neg(A \vee B)$

   (f) $A \rightarrow B \rightarrow (A \wedge B \rightarrow A)$

   (g) $((A \vee B) \wedge \neg A) \rightarrow B$

   (h) $((\neg A \vee B) \wedge (C \vee \neg B) \wedge A) \rightarrow C$

4. For each of the following English sentence write out a corresponding propositional formula. We do this by first identifying the atomic propositions (e.g. in question (a) below the two atomic propositions are "I have finish writing the exercise sheet" and "I can go to sleep") and then connecting them with the usual Boolean connectives (i.e. $\wedge$, $\vee$, $\neg$, $\rightarrow$).

   (a) If I finish writing this exercise sheet I can go to sleep.

   (b) You can take this exam if you are a linguistics student but not a first-year student.

   (c) If you get more doubles then anyone else while playing Monopoly then you will lose the game or, if you do lose, then you must have bought the most properties.

   (d) Bob is either English or Irish. If Bob is English he is British and if he is Irish he is not British. Bob is not British. Therefore Bob is Irish.

First-Order Logic: Syntax and Semantics

1. Review the lecture slides and make sure you understand how terms are constructed, how atomic formulas are constructed from terms, and how formulas are constructed. Fix a vocabulary of your choice and write out four first-order formulas over that vocabulary. If you have chosen the symbols in that vocabulary to have a particular intended meaning, can you write those formulas out in English? Take each of these formulas and circle the atomic formulas within them.

2. Review the definition of a model and the definitions of the "$\models$" relation. Let $\tau := \{R\}$, where R is a binary relation symbol. When we construct a model we assign R to some real binary relation between objects in our universe. Recall, a binary relation just relates two objects, for example "is the brother of" or "does not like" are both binary relations on the set of all people. Think of three other binary relations and in each case construct a τ-model for each such that the relation symbol R is assigned to the relation you constructed. (For example, if I construct a model in which the relation is supposed to be "is the brother of" then if x is related to y then y must also be related to x.)

3. Let $\tau := \{O, P, C, d, l, e, a, b\}$ be a vocabulary where (i) O and P are unary relation symbols meant to denote "is an occupation" and "is a person", respectively, (ii) C is a binary relation symbol such that $C(x, y)$ is meant to denote that x "is a customer of" y, (iii) d, l, e are constant symbols meant to denote "Doctor", "Lawyer", and "Engineer", respectively, and (iv) a and b are constant symbols meant to denote "Alice" and "Bob". For each of the following sentences give a corresponding first-order formula over τ with the same intended meaning:

   (a) If Alice is not a doctor she is an engineer.

   (b) Every person that is not a lawyer is an engineer.

   (c) There exists a doctor whose patient is a lawyer and that same lawyer has only engineers as clients.

   (d) there exists engineers that are also doctors.

   (e) Alice is a doctor and Bob is an engineer or the other way around, and in either case they have a customer in common.

   (f) Nobody is both a Doctor and a Lawyer, with the possible exception of Alice.

4. Let $\tau := \{R\}$, where R is a binary relations symbol. Consider the following formulas:

$$\psi_1 := \forall x \forall y (R(x, y) \rightarrow R(y, x)),$$
$$\psi_2 := (\neg \exists x R(x, y)) \leftrightarrow \forall x (\neg R(x, x)), \text{ and}$$
$$\psi_3 := (\forall x \exists y R(x, y)) \leftrightarrow (\exists y \forall x R(x, y)).$$

   (a) Which of the above formulas are tautologies? If a given formula is a tautology explain why this is the case or otherwise present a model in which the formula fails to hold.

   (b) Which of the above formulas are consistent? If a given formula is consistent present a model for which the formula holds or otherwise explain why the formula fails to hold in all models.

5. Let $\tau := \{R\}$, where R is a ternary (arity 3) relation symbol. Let

$$\psi_1 := \exists x \exists y \exists z (\neg(x = y) \wedge (\neg(y = z)) \wedge (\neg(z = x)))),$$
$$\psi_2 := \forall x \forall y \forall z [R(x, y, z) \leftrightarrow R(x, z, y)],$$
$$\psi_3 := \forall x \forall y \forall z \forall w [(R(x, y, z) \wedge R(x, z, w)) \rightarrow R(x, y, w)],$$
$$\psi_4 := \forall y \forall z \exists x R(x, y, z), \text{ and}$$
$$\psi := \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4,$$

   and let $\Gamma = \{\psi_1, \psi_2, \psi_3, \psi_4\}$.

   (a) Is the formula ψ a tautology? If not, construct a model in which it is false. Is it consistent? If so, construct a model in which it is true. Is it inconsistent? If so, explain why it fails to hold in every τ-model.

   (b) Consider the sentences:

$$\theta_1 := \forall x \forall y \forall z \forall w [R(x, y, z) \wedge R(x, z, w) \rightarrow R(x, w, y)],$$
$$\theta_2 := \forall x \forall y \forall z [R(x, y, z \rightarrow (\neg(x = y) \wedge \neg(y = z))], \text{ and}$$
$$\theta_3 := \exists x \exists y \exists z [R(x, y, z) \wedge \neg R(x, z, y)].$$

      i. Which of θ_1, θ_2, and θ_3 are entailed by Γ? If a given formula is entailed by Γ explain why and if not construct a counter example (i.e. a model in which all of the formulas in Γ holds but the given formula does not).

      ii. Is there any formula above whose negation is entailed by Γ. If so, which one?

Proof Theory: Natural Deduction

Before we start with the questions let me suggest a few online resources that might be useful for those who would like from more detail and many more examples:

- `http://logicmanual.philosophy.ox.ac.uk/vorlesung/logic6.pdf`
- `https://www.cl.cam.ac.uk/teaching/1112/LogicProof/logic-notes.pdf` (Section 3.2)

1. Review the lectures slides and convince yourself that each of the proof rules introduced are valid. To be precise, take each proof rule in turn and convince yourself that if the statements above the line all hold then the statement below the line must also hold. In doing so you are convincing yourself that the proof system is *sound* (you should also review the lecture slides to be sure you understand what it means for a proof system to be sound and complete).

2. Let U and Q be a unary relation symbols and let R and T be binary relation symbols. Let ϕ and ψ be arbitrary first-order formulas. Prove the following using natural deduction:

   (a) $\vdash \phi \to (\psi \to \phi)$,

   (b) $\vdash ((\phi \to \psi) \to \phi) \to \phi$,

   (c) $\vdash \phi \to ((\phi \wedge \psi) \vee (\phi \vee \neg\psi))$

   (d) $\exists x \forall y R(x, y) \vdash \forall x \exists y R(x, y)$,

   (e) $\vdash \exists x \forall y (U(x) \to U(y))$,

   (f) $\{\exists x U(x), \forall x (U(x)) \to Q(x)\} \vdash \exists x Q(x)$, and

   (g) $\forall x \neg \forall y (R(x, y) \to T(x, y)) \vdash \forall x \neg \forall y \neg R(x, y)$.

3. The following is an erroneous proof establishing that $\exists x(U(x) \wedge Q(x)) \vdash (\exists x U(x)) \wedge (\exists x Q(x))$:

$$\dfrac{\dfrac{\dfrac{\exists x(U(x)\wedge Q(x)) \qquad \dfrac{[U(a)\wedge Q(a)]}{U(a)}}{U(a)}}{\exists x U(x)} \qquad \dfrac{\dfrac{\exists x(U(x)\wedge Q(x)) \qquad \dfrac{[U(a)\wedge Q(a)]}{Q(a)}}{Q(a)}}{\exists x Q(x)}}{\exists x(U(x)\wedge(\exists x Q(x)))}$$

   What is the mistake in this proof? Can this proof be corrected? If so, correct it. If not, establish this by constructing a counter example (i.e. by constructing a model $\mathcal{M}$ over the vocabulary $\{U, Q\}$ such that $\mathcal{M} \models \exists x(U(x) \wedge Q(x))$ but $\mathcal{M} \not\models (\exists x U(x)) \wedge (\exists x Q(x))$). What property of the proof system guarentees that finding such a counter example suffices to establish that no such proof exists?