Symmetric Computation: Lecture 4

Anuj Dawar and Gregory Wilsenach

Department of Computer Science and Technology, University of Cambridge

ESSLLI, August 2021

Linear Programming

Linear Programming is an important algorithmic tool for solving a large variety of optimization problems.

It was shown by **(Khachiyan 1980)** that linear programming problems can be solved in polynomial time.
We have a set C of *constraints* over a set V of *variables*.
Each $c \in C$ consists of $a_c \in \mathbb{Q}^V$ and $b_c \in \mathbb{Q}$.

Feasibility Problem: Given a linear programming instance, determine if there is an $x \in \mathbb{Q}^V$ such that:

$$a_c^T x \leq b_c \quad \text{for all } c \in C$$

Optimization Problem: Given a linear programming instance and a linear *objective function* f, find a feasible point x for which $f(x)$ is maximum.

Linear Programs for Hard problems

In the 1980s there was a great deal of excitement at the discovery that *linear programming* could be done in *polynomial time*.

This raised the possibility that linear programming techniques could be used to *efficiently* solve hard problems.

Many proposals were put forth for encoding *hard* problems (such as the *Travelling Salesman Problem*) (TSP) as linear programs.

(Yannakakis 1991) proved that *any* encoding of TSP as a linear program, satisfying natural *symmetry* conditions, must have *exponential size*.

Travelling Salesman Problem

Given a set of V of n vertices and a distance matrix $C = \mathbb{Q}^{V \times V}$, find

$$\min_{\pi \in [n] \stackrel{\text{bij}}{\to} V} \sum_{i \in [n]} c_{\pi(i)\pi(i+1)} + c_{\pi(n)\pi(1)}$$

To formulate this as a *linear optimization* problem, introduce a set of variables:

$$X = \{x_{ij} \mid i, j \in V\}.$$

So, a graph is a *function* $G : X \to \{0, 1\}$.
Let $P \subseteq \{0, 1\}^X$ be the collection of simple cycles of length n.

TSP polytope

Let $\text{conv}(P) \subseteq \mathbb{Q}^X$ be the *convex hull* of P.
That is, the set of $\vec{y} \in \mathbb{Q}^X$ such that

$$\vec{y} = \sum_{\vec{x} \in P} \lambda_{\vec{x}} \vec{x} \quad \text{with } \lambda_{\vec{x}} \geq 0 \text{ and } \sum_{\vec{x} \in P} \lambda_{\vec{x}} = 1.$$

TSP: $\min \sum_{i,j \in V} c_{ij} x_{ij}$ over $\vec{x} \in P$.
This is equivalent to minimizing $\sum_{i,j \in V} c_{ij} x_{ij}$ over $\text{conv}(P)$.

We call $\text{conv}(P)$ the *TSP polytope*.

$\text{conv}(P)$ has *exponentially many facets*.

Extended Formulations

Could $\mathsf{conv}(P)$ be obtained as the *projection* of a polytope with a small number of facets?

Is there a *small* $Q \subseteq \mathbb{Q}^X \times \mathbb{Q}^Y$ such that

$$\{\vec{x} \mid \exists \vec{y}(\vec{x}, \vec{y}) \in Q\} = \mathsf{conv}(P)?$$

If a description of such a Q could be obtained in *polynomial time* in n, then $\mathsf{P} = \mathsf{NP}$.

If such a Q of *polynomial size* exists, then $\mathsf{NP} \subseteq \mathsf{P}/\mathrm{poly}$.

Also note that by adding inequalities $x \leq G(x)$ for a graph $G : X \to \{0, 1\}$, we obtain a polytope $Q_G \subseteq \mathbb{Q}^X \times \mathbb{Q}^Y$ which is *non-empty* if, and only if, G contains a Hamiltonian cycle.

Yannakakis

Say $Q \subseteq \mathbb{Q}^X \times \mathbb{Q}^Y$ is *symmetric* if for every $\pi \in S_V$, there is a $\sigma \in S_Y$ such that

$$Q^{(\pi,\sigma)} = Q$$

Here, we extend the action of π to $V \times V$, and hence to $\mathbb{Q}^X$. similarly σ to $\mathbb{Q}^Y$.

Theorem (Yannakakis)

Any symmetric $Q \subseteq \mathbb{Q}^X \times \mathbb{Q}^Y$ whose projection on $\mathbb{Q}^X$ is $\mathsf{conv}(P)$ has exponentially many facets.

This is derived from a similar lower bound for the *matching polytope*.

Matching Polytope

Fix V with $|V| = 2n$ and $X = \{x_{ij} \mid i, j \in V\}$
$M \subseteq \{0,1\}^X$ is the set of graphs that *are* perfect matchings on V.
conv(M) has an *explicit* description given by **(Edmonds)**:

$$x_{ij} \geq 0,\ \forall i, j \in V$$

$$\sum_j x_{ij} = 1\ \ \forall i \in V$$

$$\sum_{i \in S; j \notin S} x_{ij} \geq 1\ \ \forall S \subseteq V \text{ with } |S| \text{ odd},$$

This has exponentially many facets.

Lower Bounds

Theorem (Yannakakis)

Any symmetric $Q \subseteq \mathbb{Q}^X \times \mathbb{Q}^Y$ whose projection on $\mathbb{Q}^X$ is $\mathsf{conv}(M)$ has exponentially many facets.

The lower bound on the *TSP* polytope is obtained by a reduction from the lower bound on the *matching* polytope.

What if we drop the condition of *symmetry*?

A long line of work since **(Yannakakis 1991)** has looked at *relaxing* the notion of symmetry. This culminated in **(Rothvoß 2013)** showing an exponential lower bound even *without* the requirement of symmetry.

But... Linear Programming is P-complete

Any problem in P can be solved by coding it is a *linear program*.

Suppose $L \subseteq \{0,1\}^*$ is in P.
For any n, let $X = \{x_i \mid i \in [n]\}$.

There is a polytope $Q \subseteq \mathbb{Q}^X \times \mathbb{Q}^Y$ of size $\mathrm{poly}(n)$ whose projection on $\mathbb{Q}^X$ includes all points in $L \cap \{0,1\}^X$ and excludes all points in $\{0,1\}^X \setminus L$.

Note: not necessarily the *convex hull* of $L \cap \{0,1\}^X$.

Circuits to LP

Take a *circuit* C of poly-size deciding $L \cap \{0,1\}^X$.
Introduce a new variable g for each gate of C.

$$g = \neg u : \; 0 \leq g = 1 - u \leq 1$$

$$\begin{aligned} g = u \wedge v : \; & 0 \leq g \leq u \leq 1 \\ & 0 \leq g \leq v \leq 1 \\ & g \leq u + v - 1 \end{aligned}$$

and similarly for other gates.

The argument works for the non-uniform class P/poly.

Convex Hulls and Separating Polytopes

For the *matching* and *TSP* polytopes, i.e. the convex hull of solutions, we have exponential lower bounds on both symmetric (by **Yannakakis**) and general (by **Rothvoß**) versions.

For polytopes that *separate* solutions from non-solutions we have poly-size ones for *matching*, and we cannot hope for lower bounds greater than poly-size for *TSP*.

What about *symmetric* polytopes that separate solutions from non-solutions?

Symmetric Linear Programs

Fix $X = \{x_{ij} \mid i, j \in V\}$ for a fixed vertex set V.
Consider a class $\mathcal{C}$ of graphs $G : X \to \{0, 1\}$.

We say that a polytope $Q \subseteq \mathbb{Q}^X \times \mathbb{Q}^Y$ *decides* $\mathcal{C}$ if its projection on $\mathbb{Q}^X$ includes $\mathcal{C}$ and excludes its complement.

Q is *symmetric* if for each $\pi \in S_V$ there is a $\sigma \in S_Y$ such that $Q = Q^{(\pi,\sigma)}$.

The Power of Symmetric LP

In **(Atserias, D., Ochremiak 2018)** we show that the following are equivalent for a class of graphs $\mathcal{C}$.

1. $\mathcal{C}$ is decided by a family of *polynomial-size, symmetric* linear programs.
2. $\mathcal{C}$ is decided by a family of *polynomial-size, symmetric* threshold circuits.
3. $\mathcal{C}$ is decided by a family of *polynomial-size* formulas of C^k for some fixed k.

In particular, $\mathcal{C}$ must have bounded counting width.

There *are* poly-size symmetric linear programs that decide the class of graphs with *perfect matchings*.

There are *no* poly-size symmetric linear programs that decide the class of graphs with a *Hamiltonian cycle*.

Linear Programming

We can represent an instance of a linear programming feasibility problem as a *relational structure* over a suitable vocabulary.

We have a set C of *constraints* over a set V of *variables*.
Each $c \in C$ consists of $a_c \in \mathbb{Q}^V$ and $b_c \in \mathbb{Q}$.
The numbers are encoded in *binary* over an ordered set of *bit positions*.

Feasibility Problem: Given a linear programming instance, determine if there is an $x \in \mathbb{Q}^V$ such that:

$$a_c^T x \leq b_c \quad \text{for all } c \in C$$

Representing Rational Numbers

We can take the rational number

$$q = s\frac{n}{d}$$

where $s\{1, -1\}$ and $n, d \in \mathbb{N}$
to be given by a structure

$$(B, <, S, N, D)$$

where $<$ is a linear order on the domain B and S, N and D are unary relations.

$S = \emptyset$ *iff* $s = 1$ and N and D code the binary representation of n and d.

Since the domain is ordered, it is straightforward to see that arithmetic, in the form of addition and multiplication of numbers is definable in FPC

Representing Rational Vectors and Matrices

A *rational vector* indexed by a set I:

$$v : I \to \mathbb{Q}$$

is represented by a structure over domain $I \cup B$ with relations:

- $<$ an order on B;
- $S, N, D \subseteq I \times B$

Similarly, a *rational matrix* $M \in \mathbb{Q}^{I \times J}$ is given by a structure over domain $I \cup J \cup B$ with relations:

- $<$ an order on B;
- $S, N, D \subseteq I \times J \times B$

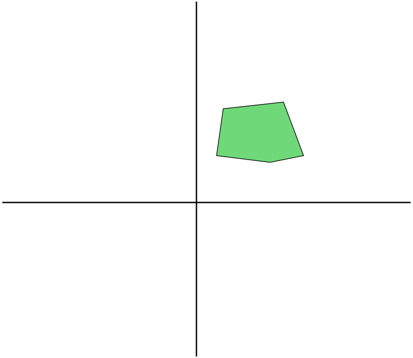
Weighted Graphs

We use a similar encoding to represent problems over *weighted graphs* where the weights may be integer or rational.

For example, a graph with vertex set V with *non-negative rational* weights might be considered as a relational structure over universe $V \cup B$ where B is bigger than the number of bits required to represent any of the rational weights and we have

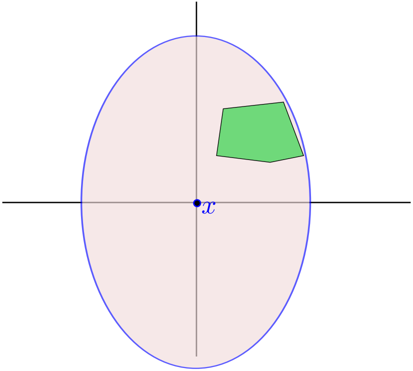
- $<$ an order on B;
- *weight relations* $W_n, W_d \subseteq V \times V \times B$

Ellipsoid Method



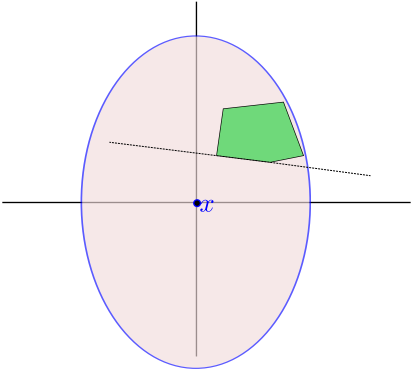
The set of constraints determines a *polytope*

Ellipsoid Method



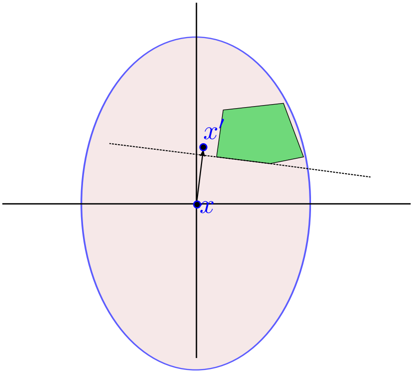
Start at the origin and calculate an *ellipsoid* enclosing it.

Ellipsoid Method



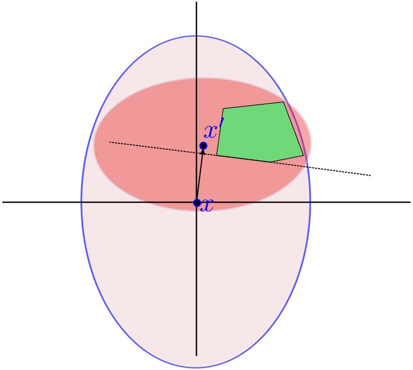
If the centre is not in the polytope, choose a constraint it *violates*.

Ellipsoid Method



Calculate a new *centre*.

Ellipsoid Method



And a new ellipsoid around the centre of at most *half* the volume.

Ellipsoid Method in FPC

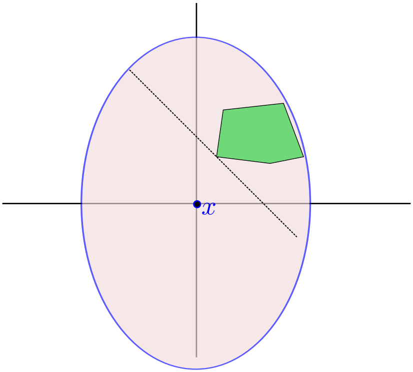
We can encode all the calculations involved in FPC.

This relies on expressing algebraic manilpulations of *unordered* matrices.

What is not obvious is how to *choose* the violated constraint on which to project.

However, the ellipsoid method works as long as we can find, at each step, some *separating hyperplane*.

Ellipsoid Method in FPC



Ellipsoid Method in FPC

We can encode all the calculations involved in FPC.

This relies on expressing algebraic manilpulations of *unordered* matrices.

What is not obvious is how to *choose* the violated constraint on which to project.

However, the ellipsoid method works as long as we can find, at each step, some *separating hyperplane*.

So, we can take:

$$(\sum_{c \in S} a_c)^T x \leq \sum_{c \in S} b_c$$

where S is the *set* of all violated constraints.

Separation Oracle

More generally, the ellipsoid method can be used, even when the *constraint matrix* is not given explicitly, as long as we can always determine a *separating hyperplane*.

In particular, the polytope represented may have *exponentially many* facets.

(Anderson, D., Holm 2015) shows that as long as the *separation oracle* can be defined in FPC, the corresponding *optimization problem* can be solved in FPC.

Representations of Polytopes

A *representation* of a class $\mathcal{P}$ of *polytopes* is a *relational vocabulary* τ along with a surjective function ν taking τ-structures to polytopes in $\mathcal{P}$, which is isomorphism invariant.

A *separation oracle* for a representation $\nu, \mathcal{P}$ is definable in FPC if there is an FPC formula that given a τ-structure $\mathbb{A}$ and a vector $v \in \mathbb{Q}^V$ either

- determines that $v \in \nu(\mathbb{A})$; or
- defines a hyperplane separating v from $\nu(\mathbb{A})$.

Folding Polytopes

We use the separation oracle to define an *ordered equivalence relation* on the set V of variables.

We also define a *projection* operation on polytopes which either

- preserves feasibility; or
- refines the equivalence relation further.

Folding and Unfolding

Suppose we have $\sigma : V \to [n]$, for $n \leq |V|$.

We say $c \in \mathbb{Q}^V$ agrees with σ, if $\sigma(u) = \sigma(v) \Rightarrow c_u = c_v$.

Fold $P \subseteq \mathbb{Q}^V$ into $P^\sigma \subseteq \mathbb{Q}^n$.

For $i \in [n]$,

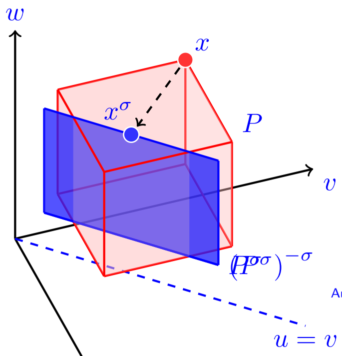
$(x^{\tilde{\sigma}})_i := \sum_{\{v \in V \mid \sigma(v)=i\}} x_v$;

$(x^\sigma)_i := \frac{(x^{\tilde{\sigma}})_i}{|\{v \in V \mid \sigma(v)=i\}|}$.

Unfold $P^\sigma \subseteq \mathbb{Q}^n$ into $(P^\sigma)^{-\sigma} \subseteq \mathbb{Q}^V$.

For $v \in V$,

$(\mathsf{x}^{-\sigma})_v := \mathsf{x}_{\sigma(v)}$.



Folding and Unfolding

Suppose we have $\sigma : V \to [n]$, for $n \leq |V|$.

We say $c \in \mathbb{Q}^V$ agrees with σ, if $\sigma(u) = \sigma(v) \Rightarrow c_u = c_v$.

Fold $P \subseteq \mathbb{Q}^V$ into $P^\sigma \subseteq \mathbb{Q}^n$.

For $i \in [n]$,

$(x^{\tilde{\sigma}})_i := \sum_{\{v \in V \mid \sigma(v)=i\}} x_v$;

$(x^\sigma)_i := \frac{(x^{\tilde{\sigma}})_i}{|\{v \in V \mid \sigma(v)=i\}|}$.

Unfold $P^\sigma \subseteq \mathbb{Q}^n$ into $(P^\sigma)^{-\sigma} \subseteq \mathbb{Q}^V$.

For $v \in V$,

$(\mathsf{x}^{-\sigma})_v := \mathsf{x}_{\sigma(v)}$.

Properties

- P^σ is a polytope.
- $\langle P^\sigma \rangle = \mathrm{poly}(\langle P \rangle)$.
- An optimum of P^σ gives an optimum of P.
- $\mathrm{SEP}(P^\sigma, \mathsf{x})$ reduces to $\mathrm{SEP}(P, \mathsf{x}^{-\sigma})$, but... only if output c agrees with σ.

Graph Matching

Recall, in a *graph* $G = (V, E)$ a matching $M \subset E$ is a set of edges such that each vertex is incident on *at most* one edge in M.

We saw that the existence of a *perfect matching* is not definable in FP.

(Blass, Gurevich, Shelah 1999) showed that for *bipartite* graphs this is definable in FPC.

They conjectured that this was *not* the case for general graphs.

We consider the more general problem of determining the *maximum weight* of a matching in a *weighted graph*:

$$G = (V, E) \quad w : E \to \mathbb{Q}_{\geq 0}$$

The Matching Polytope

(Edmonds 1965) showed that the problem of finding a *maximum weight matching* in $G = (V, E) \quad w : \mathbb{Q}^E_{\geq 0}$ can be expressed as the following linear programming problem

$$\max w^\top y \qquad \text{subject to}$$

$$\begin{aligned} & Ay \leq 1^V, \\ & y_e \geq 0, \ \ \forall e \in E, \\ & \sum_{e \in E \cap W^2} y_e \leq \frac{1}{2}(|W| - 1), \ \ \forall W \subseteq V \text{ with } |W| \text{ odd}, \end{aligned} \qquad (1)$$

Matching in FPC

A *separation oracle* for this polytope is definable by an FPC formula interpreted in the weighted graph G.

As a consequence, there is an FPC formula defining the *size* of the maximum matching in G.

Note that this does not allow us to define an *actual* matching.

Maximum Flow

MaxFlow

Given: A capacitated graph $G = (V, c)$, with $c : V \times V \to \mathbb{Q}_{\geq 0}$ and $s, t \in V$.

Determine: $f : V \times V \to \mathbb{Q}_{\geq 0}$ optimising

$$\max \sum_{v \in V} (f(v,t) - f(t,v)) \quad \text{subject to}$$

$$\sum_{v \in V} (f(v,u) - f(u,v)) = 0, \ \ \forall u \in V \setminus \{s,t\}$$

$$0 \leq f(u,v) \leq c(u,v), \ \ \forall u \neq v \in V.$$

Lemma

MaxFlow $\in$ FPC.

Proof: Polytope is explicit. Use explicit SEP with FPC reduction.

Minimum Cut

MinCut

Given: A capacitated graph $G = (V, c)$, with $c : V \times V \to \mathbb{Q}_{\geq 0}$ and $s, t \in V$.

Determine: A set $C \subseteq V$ with $s \in C$, $t \notin C$, and minimising

$$\sum_{u \in C, v \in V \setminus C} c(u, v).$$

Lemma

MinCut $\in$ FPC.

Proof:

- Compute max flow f in FPC.
- $C_f = \{v \in V \mid \text{non-0 capacity } s \rightsquigarrow v \text{ in residual graph } G|_f\}$

Lemma

C_f is independent of f. Its the canonical minimum (s, t)-cut of G.

Minimum Odd Cut

MINODDCUT

Given: A capacitated graph $G = (V, c)$, with $c : V \times V \to \mathbb{Q}_{\geq 0}$ and $|V|$ even.

Determine: A set $C \subseteq V$ with $|C|$ odd, and minimising

$$\sum_{u \in C, v \in V \setminus C} c(u, v).$$

Lemma

For some $s, t \in V$, the canonical min (s, t)-cut is a min odd cut.

Proof Idea: Collapse sets of vertices while preserving existence of some min odd cut.

Lemma

FPC *can define a small set of min odd cuts.*

Matching

b-MATCHING

Given: $G = (V, E)$ and $A \in \{0,1\}^{V \times E}, b \in \mathbb{N}^V, c \in \mathbb{Q}^E_{\geq 0}$. **Determine:** $y \in \mathbb{N}^E_{\geq 0}$ optimising

$$\max c^\top y \quad \text{subject to} \quad Ay \leq b, y \geq 0^E.$$

Specialises to MAXMATCHING when $b = 1^V, c = 1^E$.

Relax LP (i.e., $y \in \mathbb{Q}^E_{\geq 0}$) and add constraints consistent with integral solutions:

$$y(W) \leq \frac{1}{2}(b(W) - 1), \forall W \subseteq V \text{ with } b(W) \text{ odd.}$$

where $y(W) = \sum_{e \in E, e \subseteq W} y_e$ and $b(W) = \sum_{v \in W} b_v$.

Theorem (Edmonds '65)

The extremal points of the relaxed LP are integral.

Matching, contd.

Lemma (Padberg-Rao '82)

Given $y \in \mathbb{Q}^E_{\geq 0}$. There is exists a capacitated graph H such that y violates an odd set constraint iff H has a min odd cut of value < 1.

- FPC can define H from y.
- FPC can define a small set of min odd cuts of H.
- FPC can define a small set of violated odd set constraints.
- FPC can define a canonical violated constraint (by linearity).

Lemma

There is an FPC interpretation $\mathrm{fin}[\tau_{match} \uplus \tau_{vec}] \to \mathrm{fin}[\tau_{vec}]$ expressing the separation problem for b-MATCHING polytopes with respect to their natural representation as τ_{match}-structures.

Symmetric LPs

For $s = O(2^{n^{1-\epsilon}}), \epsilon > 0$:

1. a symmetric circuit of size s translates to a symmetric LP of size $\text{poly}(s)$; and
2. a symmetric LP of size s translates to a formula of C^k with $k = O(\frac{\log s}{\log n})$.

So, *polynomial-size* families of symmetric circuits and symmetric LPs are *equivalent*.

Translations

The translation from circuits to linear programs starts from the one given by **Yannakakis**, but we have to

- account for *majority* (or *threshold*) gates; and
- preserve *symmetry*

To achieve these two feats simultaneously requires some work.

Linear Programs to Formulas

Starting with a linear program P defining a *symmetric polytope* $Q \subseteq \mathbb{Q}^X \times \mathbb{Q}^Y$, where $X = [n] \times [n]$, we can:

Partition Y into *orbits* under the induced action of S_n;

replace the orbits with single variables by *linearity*.

This gives us an equivalent *reduced* linear program $\hat{P}$ that is *rigid*.

We do not know if this can be done in polynomial-time, so we can't guarantee we get a uniform family.

Evaluating Symmetric LPs

We have $\hat{P}$, which defines a *rigid* symmetric polytope $Q \subseteq \mathbb{Q}^X \times \mathbb{Q}^{\hat{Y}}$, where $X = [n] \times [n]$

And a graph G on n vertices.

Any bijection $\beta : V(G) \to [n]$ gives a polytope $Q_\beta \subseteq \mathbb{Q}^{\hat{Y}}$. By symmetry, these are all the same up to a permutation of $\hat{Y}$.

We show that we can obtain an LP equivalent to Q_β by a C^k-*interpretation* (for $k = \frac{\log s}{\log n}$) from the graph G, with advice $\hat{P}$.

Supports

We can show that, under the action of S_n on $\hat{P}$, the *stabilizer* of each variable in Y and each constraint in $\hat{P}$ has a *support* of size $k = O(\frac{\log s}{\log n})$.

Theorem

If $n > 8$, $1 \leq k \leq n/4$, and G is a subgroup of S_n with $[S_n : G] < \binom{n}{k}$, then there is a set $S \subseteq [n]$ with $|S| < k$ such that $A_{(S)} \leq G$.

Alternating Groups

To show that we can replace the alternating group by the *symmetric group*, we cannot rely on an induction on depth, as we did with circuits.

Instead, we show that if some variable in Y does *not* have small support, we can construct a small (i.e. size $\text{poly}(s)$) graph whose *automorphism group* is isomorphic to $A_{(S)}$.

Theorem

If $n > 22$, then the number of vertices of any graph whose full automorphism group is isomorphic to A_n is at least $1/2\binom{n}{\lfloor n/2 \rfloor} \sim 2^n/\sqrt{2\pi n}$.