Symmetric Computation: Lecture 3

Anuj Dawar and Gregory Wilsenach

Department of Computer Science and Technology, University of Cambridge

ESSLLI, August 2021

Cai-Fürer-Immerman Graphs

Cai-Fürer-Immerman show that there is a polynomial-time graph property that is not in FPC by constructing a sequence of pairs of graphs $G_k, H_k (k \in \omega)$ such that:

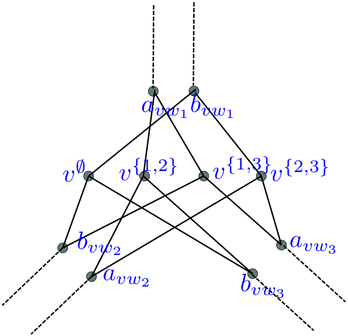
- $G_k \equiv^{C^k} H_k$ for all k.
- There is a polynomial time decidable class of graphs that includes all G_k and excludes all H_k.

In particular, the first point shows that $\equiv^{C^k}$ (for any fixed k) does not capture isomorphism everywhere

Constructing G_k and H_k

Given any graph G, we can define a graph X_G by replacing every edge with a pair of edges, and every vertex with a gadget.

The picture shows the gadget for a vertex v that is adjacent in G to vertices w_1, w_2 and w_3.
The vertex v^S is adjacent to $a_{vw_i} (i \in S)$ and $b_{vw_i} (i \notin S)$ and there is one vertex for all *even size* S.
The graph $\tilde{X}_G$ is like X_G except that at *one vertex* v, we include v^S for *odd size* S.



Properties

If G is *connected* and has *treewidth* at least k, then:

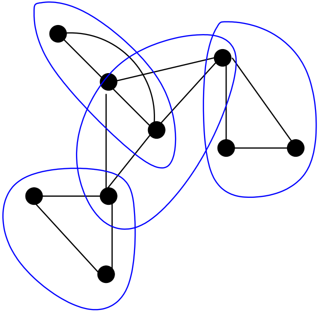
1. $X_G \ncong \tilde{X}_G$; and
2. $X_G \equiv^{C^k} \tilde{X}_G$.

(1) allows us to construct a polynomial time property separating X_G and $\tilde{X}_G$.
(2) is proved by a game argument.

> *The original proof of* **(Cai, Fürer, Immerman)** *relied on the existence of balanced separators in* G*. The characterisation in terms of treewidth is from* **(D., Richerby 07)***.*

TreeWidth

The *treewidth* of a graph is a measure of how tree-like the graph is. A graph has treewidth k if it can be covered by subgraphs of at most $k+1$ nodes in a tree-like fashion.



TreeWidth

Formal Definition:
For a graph $G = (V, E)$, a *tree decomposition* of G is a relation $D \subset V \times T$ with a tree T such that:

- for each $v \in V$, the set $\{t \mid (v, t) \in D\}$ forms a connected subtree of T; and
- for each edge $(u, v) \in E$, there is a $t \in T$ such that $(u, t), (v, t) \in D$.

We call $\beta(t) := \{v \mid (v, t) \in D\}$ the *bag* at t.

The *treewidth* of G is the least k such that there is a tree T and a tree-decomposition $D \subset V \times T$ such that for each $t \in T$,

$$|\{v \in V \mid (v, t) \in D\}| \leq k + 1.$$

Cops and Robbers

A game played on an undirected graph $G = (V, E)$ between a player controlling k cops and another player in charge of a robber.

At any point, the cops are sitting on a set $X \subseteq V$ of the nodes and the robber on a node $r \in V$.
A move consists in the cop player removing some cops from $X' \subseteq X$ nodes and announcing a new position Y for them. The robber responds by moving along a path from r to some node s such that the path does not go through $X \setminus X'$.
The new position is $(X \setminus X') \cup Y$ and s. If a cop and the robber are on the same node, the robber is caught and the game ends.

Strategies and Decompositions

Theorem (Seymour and Thomas 93):
There is a winning strategy for the *cop player* with k cops on a graph G if, and only if, the tree-width of G is at most $k-1$.

It is not difficult to construct, from a tree decomposition of width k, a winning strategy for $k+1$ cops.

Somewhat more involved to show that a winning strategy yields a decomposition.

Cops and Robbers on the Grid

If G is the $k \times k$ toroidal grid, than the *robber* has a winning strategy in the k-*cops and robbers* game played on G.

To show this, we note that for any set X of at most k vertices, the graph $G \setminus X$ contains a connected component with at least half the vertices of G.

If all vertices in X are in distinct rows then $G \setminus X$ is connected. Otherwise, $G \setminus X$ contains an entire row and in its connected component there are at least $k - 1$ vertices from at least $k/2$ columns.

Robber's strategy is to stay in the large component.

Cops, Robbers and Bijections

We use this to construct a winning strategy for Duplicator in the k-pebble bijection game on X_G and $\tilde{X}_G$.

- A bijection $h : X_G \to \tilde{X}_G$ is *good bar* v if it is an isomorphism everywhere except at the vertices v^S.
- If h is good bar v and there is a path from v to u, then there is a bijection h' that is good bar u such that h and h' differ only at vertices corresponding to the path from v to u.
- Duplicator plays bijections that are good bar v, where v is the *robber position* in G when the cop position is given by the currently pebbled elements.

Counting Width

For any class of structures $\mathcal{C}$, we define its *counting width* $\nu_{\mathcal{C}} : \mathbb{N} \to \mathbb{N}$ so that

> $\nu_{\mathcal{C}}(n)$ *is the least* k *such that* $\mathcal{C}$ *restricted to structures with at most* n *elements is closed under* $\equiv^{C^k}$.

Every class in FPC has counting width bounded by a *constant*.

The *CFI* construction based on *grids* gives a class of graphs in P that has counting width $\Omega(\sqrt{n})$.

This can be improved to $\Omega(n)$ by taking, instead of grids, *expander graphs*.

Interpretations

Given two relational signatures σ and τ, where $\tau = \langle R_1, \ldots, R_r \rangle$, and arity of R_i is n_i

A *first-order interpretation of τ in σ* is a sequence:

$$\langle \pi_U, \pi_1, \ldots, \pi_r \rangle$$

of first-order σ-formulas, such that, for some d:

- the free variables of π_U are among $x_1, \ldots, x_d$,
- and the free variables of π_i (for each i) are among $x_1, \ldots, x_{d \cdot n_i}$.

d is the dimension of the interpretation.

Interpretations II

An interpretation of τ in σ maps σ-structures to τ-structures.

If $\mathbb{A}$ is a σ-structure with universe A, then
$\pi(\mathbb{A})$ is a structure $(B, R_1, \ldots, R_r)$ with

- $B \subseteq A^d$ is the relation defined by π_U.
- for each i, R_i is the relation on B defined by π_i.

An FO *reduction* of a class of structures $\mathcal{C}$ to a class $\mathcal{D}$ is a single FO interpretation θ such that $\mathbb{A} \in \mathcal{C}$ if, and only if, $\theta(\mathbb{A}) \in \mathcal{D}$.
We write $\mathcal{C} \leq_{\mathsf{FO}} \mathcal{D}$.

FPC-Reductions

More generally, we can defined reductions in any logic, *e.g.* FPC.

If $\mathcal{C} \leq_{\mathsf{FPC}} \mathcal{D}$ then

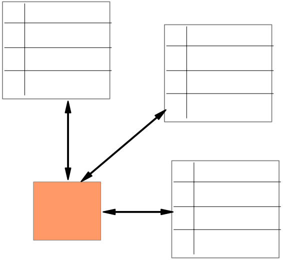
$$\nu_{\mathcal{D}} = \Omega(\nu_{\mathcal{C}}^{1/d}).$$

If the reduction takes $\mathcal{C}$-instances to $\mathcal{D}$-instances of *linear size*, then

$$\nu_{\mathcal{D}} = \Omega(\nu_{\mathcal{C}}).$$

By means of reductions, we can estalish *3-Sat*, *XOR-Sat*, *3-Colourability*, *Hamiltonicity* all have counting width $\Omega(n)$.

Relational Machines



Input: A relational database

Store: relational and numerical registers

Operations: *join, projection, complementation, counting*

Properites expressible in FPC are *exactly* those decidable by such a machine in *polynomial time*.

Relational Machines - Formally

Fix a relational vocabulary $\sigma = (R_1, \ldots, R_m)$.

The relational machine M has:

- fixed relational registers: $R_1, \ldots, R_m$;
- a fixed number of *variable* relational registers $P_1, \ldots, P_s$ each with fixed arity; and
- a fixed number of *numerical registers*: $c_1, \ldots, c_t$.

Relational Machines - Formally

The program of M is composed of *atomic actions*

$P_i := R_j$; $P_i := P_j \cup P_k$; $P_i := \pi_{a_1,\dots,a_k} P_j$;
$P_i := P_j \bowtie_{a_1,\dots,a_k} P_k$; $P_i := \overline{P_j}$.
$c_i := c_i + 1$;
$c_i := \#P_j$

and control commands:

if $c_i = 0$ *then* p *else* $c_i := c_i - 1$
while $c_1 \neq 0$ p

Exercise: Show that a class of relational structures is accepted by a *polynomial-time* relational machine if, and only if, it is definable in FPC.

Circuits

A circuit C is a *directed acyclic graph* with:

- source nodes (called *inputs*) labelled $x_1, \ldots, x_n$;
- any other node (called a *gate*) with k incoming edges is labelled by a Boolean function $g : \{0,1\}^k \to \{0,1\}$ from some fixed basis (*e.g.* AND/OR/NOT);
- some gates designated as *outputs*, $y_1, \ldots, y_m$.

C computes a function $f_C : \{0,1\}^n \to \{0,1\}^m$ as expected.

Circuit Complexity

A *language* $L \subseteq \{0,1\}^*$ can be described by a family of *Boolean functions*:

$$(f_n)_{n\in\omega} : \{0,1\}^n \to \{0,1\}.$$

Each f_n may be given by a *circuit* C_n made up of AND/OR/NOT gates, with n inputs and one output.

If the size of C_n is bounded by a polynomial in n, the language L is in the class P/poly.

If, in addition, the function $n \mapsto C_n$ is computable in polynomial time, L is in P.

Circuit Complexity Classes

For the definition of P/poly and P, it makes no difference if the circuits only use $\{\mathsf{AND}, \mathsf{OR}, \mathsf{NOT}\}$ or a richer basis with *ubounded fan-in; threshold; or counting gates*.

However,

> *AC_0 — languages accepted by bounded-depth, polynomial-size families of circuits with unbounded fan-in AND and OR gates and NOT gates;*

and

> *TC_0 — languages accepted by bounded-depth, polynomial-size families of circuits with unbounded fan-in AND and OR and threshold gates and NOT gates;*

are different.

A *threshold gate* $\mathsf{Th}_t^k : \{0,1\}^k \to \{0,1\}$ evaluates to 1 iff at least t of the inputs are 1.

Invariant Circuits

Instead of a language $L \subseteq \{0,1\}^*$, consider a class $\mathcal{C}$ of directed *graphs*. This can be given by a family of *Boolean functions*:

$$(f_n)_{n\in\omega} : \{0,1\}^{n^2} \to \{0,1\}.$$

A graph on vertices $\{1,\ldots,n\}$ has n^2 *potential* edges.
So the graph can be treated as a string in $\{0,1\}^{n^2}$.

Since $\mathcal{C}$ is closed under isomorphisms, each function f_n is invariant under the natural action of S_n on n^2.

We call such functions *graph invariant*.

Symmetric Circuits

More generally, for any *relational vocabulary* τ, let

$$\tau(n) = \sum_{R \in \tau} n^{\mathsf{arity}(R)}$$

We take an encoding of n-element τ-structures as strings in $\{0,1\}^{\tau(n)}$ and this determines an action of S_n on such strings.

A function $f : \{0,1\}^{\tau(n)} \to \{0,1\}$ is τ*-invariant* if it is invariant under this action.

We say that a circuit C with inputs labelled by $\tau(n)$ is *symmetric* if every $\pi \in S_n$ acting on the inputs of C can be extended to an *automorphism* of C.

Every symmetric circuit computes an invariant function, but the converse is false.

Logic and Circuits

Any formula of φ *first-order logic* translates into a uniform family of circuits C_n

> *For each subformula $\psi(\overline{x})$ and each assignment $\overline{a}$ of values to the free variables, we have a gate.*
> *Existential quantifiers translate to big disjunctions, etc.*

The circuit C_n is:

- of *constant* depth (given by the depth of φ);
- of size at mose $c \cdot n^k$ where c is the number of subformulas of φ and k is the *maximum number of free variables* in any subformula of φ.
- *symmetric* by the action of $\pi \in S_n$ that takes $\psi[\overline{a}]$ to $\psi[\pi(\overline{a})]$.

FP and Circuits

For every sentence φ of FP there is a k such that for every n, there is a formula φ_n of L^k that is equivalent to φ on all graphs with at most n vertices.

The formula φ_n has

- *depth* n^c for some constant c;
- at most k free variables in each sub-formula for some constant k.

It follows that every graph property definable in FP is given by a family of *polynomial-size, symmetric* circuits.

FPC and Circuits

For every sentence φ of FP there is a k such that for every n, there is a formula φ_n of C^k that is equivalent to φ on all graphs with at most n vertices.

The formula φ_n has

- *depth* n^c for some constant c;
- at most k free variables in each sub-formula for some constant k.

It follows that every graph property definable in FP is given by a family of *polynomial-size, symmetric* circuits in a basis with *threshold gates*.

Note: we could also alternatively take a basis with *majority* gates.

Relating Circuits and Logic

The following are established in **(Anderson, D. 2017)**:

Theorem
A class of graphs is accepted by a P-uniform, polynomial-size, symmetric family of Boolean circuits if, and only if, it is definable by an FP formula interpreted in $G \uplus ([n], <)$.

Theorem
A class of graphs is accepted by a P-uniform, polynomial-size, symmetric family of threshold circuits if, and only if, it is definable in FPC.

Some Consequences

We get a natural and purely circuit-based characterisation of FPC definability.

Inexpressibility results for FP and FPC yield lower bound results against natural circuit classes.

- There is no polynomial-size family of symmetric Boolean circuits deciding if an n vertex graph has an even number of edges.
- Polynomial-size families of uniform symmetric *threshold circuits* are more powerful than Boolean circuits.
- Invariant circuits *cannot* be translated into equivalent symmetric threshold circuits, with only polynomial blow-up.

Automorphisms of Symmetric Circuits

For a symmetric circuit C_n we can assume *w.l.o.g.* that the automorphism group is the symmetric group S_n acting in the natural way.

That is:

- Each $\pi \in S_n$ gives rise to a *non-trivial* automorphism of C_n (otherwise C_n would compute a constant function).
- There are no *non-trivial* automorphisms of C_n that fix all the inputs (otherwise there is redundancy in C_n that can be eliminated).

We call a circuit satisfying these conditions *rigid*.

By abuse of notation, we use $\pi \in S_n$ both for permutations of $[n]$ and automorphisms of C_n.

Stabilizers

For a gate g in C_n, $\mathrm{Stab}(g)$ denotes the *stabilizer group of* g, i.e. the *subgroup* of S_n consisting:

$$\mathrm{Stab}(g) = \{\pi \in S_n \mid \pi(g) = g\}.$$

The *orbit* of g is the set of gates $\{h \mid \pi(g) = h \text{ for some } \pi \in S_n\}$

By the *orbit-stabilizer* theorem, there is one gate in the orbit of g for each *co-set* of $\mathrm{Stab}(g)$ in S_n.
Thus the size of the *orbit* of g in C_n is $[S_n : \mathrm{Stab}(g)] = \frac{n!}{|\mathrm{Stab}(g)|}$.

So, an upper bound on $\mathrm{Stab}(g)$ gives us a lower bound on the orbit of g.

Conversely, knowing that the orbit of g is at most polynomial in n tells us that $\mathrm{Stab}(g)$ is *big*.

Supports

For a group $G \subseteq S_n$, we say that a set $X \subseteq [n]$ is a *support* of G if

For every $\pi \in S_n$, *if* $\pi(x) = x$ *for all* $x \in X$, *then* $\pi \in G$.

In other words, G contains all permutations of $[n] \setminus X$.
So, if $|X| = k$, $[S_n : G]$ is at most $\frac{n!}{(n-k)!} \leq n^k$.

Groups with small support are *big*.

The converse is clearly false since $[S_n : A_n] = 2$, but A_n has no support of size less than $n - 1$.

Note: For the family of circuits $(C_n)_{n \in \omega}$ obtained from an FPC formula there is a constant k such that all gates in each C_n have a support of size at most k.

Support Theorem

In *polynomial size* symmetric circuits, all gates have (stabilizer groups with) *small* support:

Theorem

For any polynomial p, there is a k such that for all sufficiently large n, if C is a symmetric circuit on $[n]$ of size at most $p(n)$, then every gate in C has a support of size at most k.

The general form of the support theorem in **(Anderson, D. 2017)** gives bounds on the size of supports in *sub-exponential* circuits.

Alternating Supports

Groups with small support are *big*.

The converse is clearly false since $[S_n : A_n] = 2$, but A_n has no support of size less than $n - 1$.

In a sense, the alternating group is the *only* exception, due to a standard result from permutation group theory.

Theorem

If $n > 8$, $1 \leq k \leq n/4$, and G is a subgroup of S_n with $[S_n : G] < \binom{n}{k}$, then there is a set $X \subseteq [n]$ with $|X| < k$ such that $A_{(X)} \leq G$.

where $A_{(X)}$ denotes the group $\{\pi \in A_n : \pi(i) = i \text{ for all } i \in X\}$

Supports of Gates

Theorem

If $n > 8$ *and* $1 \leq k \leq n/4$*, and* G *is a subgroup of* S_n *with* $[S_n : G] < \binom{n}{k}$*, then there is a set* $X \subseteq [n]$ *with* $|X| < k$ *such that* $A_{(X)} \leq G$*.*

If $(C_n)_{n\in\omega}$ is a family of *symmetric* circuits of size n^k, then for all sufficiently large n and gates g in C_n, there is a set $X \subseteq [n]$ with $|X| \leq k$ such that $A_{(X)} \leq \mathrm{Stab}(g)$.

It follows that if *any odd* permutation of $[n]$ that fixes X pointwise, also fixes g, then $S_{(X)} \leq \mathrm{Stab}(g)$, so X is a support of g.

where $S_{(X)}$ denotes the group $\{\pi \in S_n : \pi(i) = i \text{ for all } i \in X\}$

Supports of Gates

Some odd permutation of $[n]$ that fixes X pointwise, also fixes g. $(*)$

We can prove, by induction on the depth of g in the circuit C_n that this must be the case.

It is clearly true for input gates $R(\overline{a})$, as any permutation that fixes $\overline{a}$ fixes the gate.

Let g be a gate such that $(*)$ is true for all gates that are inputs to g.

Since g computes a *symmetric* Boolean function, and C_n is *rigid*, any $\pi \in S_n$ that fixes the inputs to g *setwise*, fixes g.
Let H be the set of inputs to g. By induction hypothesis, they all have a support of size at most k

Supports of Gates

Some odd permutation of $[n]$ that fixes X pointwise, also fixes g. $(*)$

Let g be a gate such that $(*)$ is true for all gates in H, but false for g

For any $i, j \in [n] \setminus X$, the permutation $(i\ j)$ moves g, so moves some $h \in H$.

$$[n] \setminus X \subseteq \bigcup_{h \in H} \mathrm{sp}(h)$$

We can then find $\frac{n-k}{k}$ elements of H with *pairwise disjoint* support.

This gives us $\frac{n-k}{k}$ distinct permutations $(i\ j)$ which we can *independently* combine to show that the orbit of g has size at least $2^{(n-k)/k}$.

Support Theorem

In *polynomial size* symmetric circuits, all gates have (stabilizer groups with) *small* support:

Theorem

For any $0 < \epsilon < 1)$, *if* C *is a symmetric circuit over* $[n]$ *of size* s *for large enough* n *and* $s \leq 2^{n^{1-\epsilon}}$. *Then every gate* g *of* C *has a support of size at most* $O(\frac{\log s}{\log n})$.

We can push this to exponential bounds: if $s = 2^{o(n)}$, then the family of circuits C_n has supports of size $o(n)$.

We write $\mathrm{sp}(g)$ for the small support of g given by this theorem and note that it can be computed in polynomial time from a symmetric circuit C.

Translating Symmetric Circuits to Formulas

Given a polynomial-time function $n \mapsto C_n$ that generates symmetric circuits:

1. There are formulas of FP interpreted on $([n], <)$ that define the structure C_n.
2. We can also compute in polynomial time (and therefore in FP on $([n], <)$) $\mathrm{sp}(g)$ for each gate g.
3. For an input structure $\mathbb{A}$ and an assignment $\gamma : [n] \to \mathbb{A}$ of the inputs of C_n to elements of $\mathbb{A}$, whether g is made true depends only on $\gamma(\mathrm{sp}(g))$.
4. We define, by induction on the structure of C_n, the set of tuples $\Gamma(g) \subseteq \mathbb{A}^{\mathrm{sp}(g)}$ that represent assignments γ making g true.
5. This inductive definition can be turned into a formula (of FP for a Boolean circuit, of FPC for one with threshold gates.)

Circuits and Pebble Games

We can use *bijection games* and the *support theorem* to establish lower bounds for symmetric circuits.

The key is the following connection.

> *If C is a symmetric circuit on n-element structures such that every gate of C has a support of size at most k, and $\mathbb{A}$ and $\mathbb{B}$ are structures such that $\mathbb{A} \equiv^{C^{2k}} \mathbb{B}$ then:*
>
> *C accepts $\mathbb{A}$ if, and only if, C accepts $\mathbb{B}$.*

This can be proved by showing that if C distinguishes $\mathbb{A}$ from $\mathbb{B}$, then it provides a *winning strategy* for *Spoiler* in the $2k$-pebble bijection game.

Proof Sketch

Show that if C accepts $\mathbb{A}$ and rejects $\mathbb{B}$, then *Spoiler* has a winning strategy in the $2k$-pebble bijection game played on $\mathbb{A}$ and $\mathbb{B}$. The number of moves needed is at most kd, where d is the *depth* of C.

Spoiler fixes a *bijection* $\alpha : A \to [n]$.

Show by induction that, while playing the *bijection game Spoiler* can maintain a pointer to a gate g of C and the following invariants for the game position $(\overline{u}, \overline{v})$:

- $\alpha(\overline{u})$ includes the support of g.
- For any bijection $\beta : B \to [n]$ such that $\beta^{-1}\alpha(\overline{u}) = \overline{v}$:

$$C_g(\alpha(\mathbb{A})) \neq C_g(\beta(\mathbb{B})).$$

Proof Sketch – 2

Base Case:
If g is the *output gate*, by assumption $C_g(\alpha(\mathbb{A})) = 1$ and for any β, $C_g(\beta(\mathbb{B})) = 0$

Induction Step:
While keeping pebbles on the support of g, *Spoiler* moves the other k pebbles to the support of a *child* h of g.
At each move, *Duplicator* plays a bijection $\gamma : A \to B$ such that $\gamma(\overline{u}) = \overline{v}$.
Thus, $C_g(\alpha(\mathbb{A})) \neq C_g(\alpha\gamma^{-1}(\mathbb{B}))$, and there is an h for which

$$C_h(\alpha(\mathbb{A})) \neq C_h(\alpha\gamma^{-1}(\mathbb{B}))$$

Circuits and Pebble Games

If C is a symmetric circuit on n-vertex graphs such that every gate of C has a support of size at most k, and $\mathbb{A}$ and $\mathbb{B}$ are graphs such that $\mathbb{A} \equiv^{C^{2k}} \mathbb{B}$ then:

C accepts $\mathbb{A}$ if, and only if, C accepts $\mathbb{B}$.

As a consequence, if $\mathcal{C}$ is a class of structures of *counting width* $k : \mathbb{N} \to \mathbb{N}$, then any family of symmetric circuits accepting $\mathcal{C}$ has size $\Omega(n^k)$.

at least for $k \leq \frac{n}{\log n}$.