Session 2: Discussion Points

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In this discussion session we will review some background on selected topics and discuss the exercises below. The topics to be discussed include:

1. The proofs of Fagin's theorem and the Immerman-Vardi Theorem;
2. more on bounded-variable fragments of first-order logic with and without counting quantifiers; and
3. Weisfeiler-Leman equivalences.

Exercises

1. It is asserted on slide 10 of the lectures that (i) if there is a cononical string representation that can be constructed in polynomial time then there is a logic for P (ii) and if not then P $\neq$ NP. Prove both of these claims.
2. It is asserted on slide 11 of the lectures that for any FO formula $\psi(R, \vec{x})$ which uses at most k variables and where R is a relation variable, the corresponding inflationary fixed-point is definable in L^{2k}. Prove this. What happens to the formula size as a result of this translation? What about the size of the circuit corresponding to this formula?
3. With the previous question in mind, prove that any formula of FP may be translated to a family of symmetric circuits of polynomial size and depth.
4. Prove that for any $\phi \in C^k$ there exists $m \in \mathbb{N}$ and $\psi \in L^m$ such that $\mathcal{A} \models \phi$ if, and only if, $\mathcal{A} \models \psi$ for all structures $\mathcal{A}$. What is the relationship between k and m?
5. Give an example of a graph property that is not definable in L^3 and prove this using pebble games.
6. Construct a family $(G_i, H_i)_{i \in \mathbb{N}}$ of pairs of non-isomorphic graphs such that for all $i \in \mathbb{N}$, $G_i \not\equiv^{L^i} H_i$. Derive from this a class of graphs not definable in L^k for any k.